On Misspecified *ARMA* Model Fittings to Exponential Processes

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Abstract. We investigate some properties on a misspecified Gaussian ARMA(p,q) model fitting to Exponential processes with order 2 (abbreviated to EX(2) process). Our main purposes are to get to know a number of globally and locally maximal points of the conditional quasi-maximum (Gaussian) likelihood function when the sample size is sufficiently large. We shall derive a mathematical form of the conditional quasi-maximum likelihood function of the ARMA(1,1) model parameters, and investigate the conditions for EX(2) parameters on which the ARMA(1,1) conditional Gaussian likelihood function has more than one locally maximal points in the stationary and invertible ARMA(1,1) parameter space.

Keywords: *ARMA(1,1) model fitting, EX(2) process, misspecification, conditional Gaussian likelihood function.*

1. Introduction

In time series analysis, some suitable linear models are fitting to a given time series data to predict a future value by the model. But, in general, we do not know the true model for the series. If a fitted model is wrong, what kind of problem arises? When we fit an MA(1) model to some special time series data which is not followed by MA(1) process, it is known that the MA(1) parameter does not have an unique Gaussian quasi-maximum likelihood estimator. Tanaka and Huzii [8] investigated the conditions of AR(2) parameters on which the MA(1) quasi-likelihood function has more than one local maximal points in the invertible parameter space (-1, 1). Furthermore, Tanaka and Aoki [7] gave the region for the AR(2) parameters on which the MA(1) quasi-likelihood function has more than one local maximal points in the parameter space. In this case, maximizing the likelihood function is equivalent to minimizing the following function S(x; a, b) when the data length is sufficiently large (see [1], [8]). Here x is an MA(1) parameter and a and b are AR(2) parameters.

$$S(x;\, a,\, b) = \frac{1 + b - a\,(1-b)\,x - b\,(1+b)\,x^2}{(1-b)\left(1 - a^2 + 2\,b + b^2\right)\left(1 - x^2\right)\left(1 + a\,x + b\,x^2\right)}. \tag{1}$$

From Tanaka and Huzii [8], there are two minimal points of the function *S(x;a,b)*. For example, in the case of an AR(2) process with a = -0.1, b = 0.8, the function *S(x;a,b)* has a graph shown in figure 1.

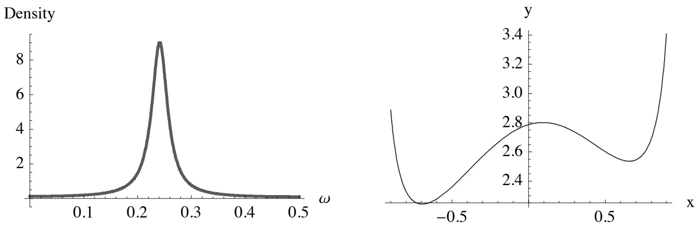

Figure 1. Graphs of *the spectrum and the S(x;a,b) for the AR(2) process* with a = -0.1, b = 0.8.

Applying a stationary ARMA model to time series data in actual time series data analysis, there is a possibility that two or more candidates for the model parameters exist, and then we cannot determine the parameters of the model well. We also know that the ARMA(1,1) model seems to be more sensitive than MA(1) model about incorrect discernment. Therefore, if such a phenomenon appears in the parameter estimation for an ARMA model fitting, the applied model must be different from a true (or proper) model, and then the model should be exchanged immediately.

Our main object for our researches is to know what kind of misspecification is fatal in the time series ARMA model fitting. How many estimated model parameters are there at most in the misspecified ARMA model fittings? In this paper we shall employ an Exponential process (see Nakatuka [6]) which is quite different from ARMA process, and we consider the problem for ARMA model fittings to this process. We shall derive a mathematical form of the conditional quasi-maximum likelihood function of the ARMA(1,1) model parameters when the sample size tends to infinity. It is seen from the numerical analysis study that, similar to the MA(1) model fitting to AR(2) process, there exist two MA(1) model parameters in the MA(1) model fitting to EX(2) process.

It is supported by the computer software *Mathematica V10.3* and its application *Time Series Pack for Mathematica* ([4]).

2. ARMA model and Exponential process

Let {Z(t)} be a weakly stationary process with $EZ(t) = 0$. {Z(t)} is said to satisfy an auto regressive moving average model of order p and q (abbreviated to ARMA(p, q) model) if {Z(t)} is expressed as

$$(1 - a_1 B - \dots - a_p B^p)\, Z(t) = (1 + b_1 B + \dots + b_q B^q)\, e(t), \tag{2}$$

where {e(t)}, t being an integer, consists of independently and identically distributed random variables with $E[e(t)] = 0$, $E\left[e(t)^2\right] = \sigma^2$, the a_p's and b_q's are constants which are independent of t, and B is the usual back shift operator such that $B[e(t)] = e(t-1)$ and $B^k[e(t)] = B[B^{k-1}[e(t)]]$ for k =1,2,.. (see, for example, [2], [3]). In this case we also say that the process {Z(t)} is ARMA(p,q) process.

Let

$$\phi(B) = 1 - a_1 B - \dots - a_p B^p = \prod_{k=1}^{p} (1 - \phi_k B), \tag{3}$$

$$\theta(B) = 1 + b_1 B + \dots + b_q B^q = \prod_{k=1}^{q} (1 - \theta_k B). \tag{4}$$

In our model fitting, it is assumed that $|\phi_h| < 1$, $|\theta_k| \leq 1$ for all h = 1, 2,$\cdots$, p, and k = 1, 2,$\cdots$, q. Let $\Theta = (\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q)$ be a (p+q)-dimensional unknown parameter, and let $\{F_k(\Theta)\}$ be a sequence of functions of Θ, which are defined in the following way. For t > 0,

$$e(t) = \left\{\prod_{k=1}^{p} (1 - \phi_k B) \prod_{k=1}^{q} (1 - \theta_k B)^{-1}\right\} Z(t) = \left\{\sum_{k=1}^{\infty} F_k(\Theta)\, B^k\right\} Z(t). \tag{5}$$

For evaluating the asymptotic properties of the conditional quasi-maximum likelihood estimators of Θ when the sample size tends to infinity, we should attend to a function

$$S_{p,q}(\Theta) = E\left[e(t)^2\right]$$

$$= \int_{-1/2}^{1/2} \frac{|\prod_{k=1}^{p} [1 - \phi_k \exp(-2\pi i \omega)]|^2}{|\prod_{j=1}^{q} [1 - \theta_j \exp(-2\pi i \omega)]|^2} f_Z(\omega)\, d\omega. \tag{6}$$

Furthermore, the value $\hat{\Theta}$ which minimizes $S_{p,q}(\Theta)$ with respect to Θ should be obtained (see Tanaka and Huzii [8] and also Huzii [4]). The spectrum of an ARMA(p,q) process $f_Z(\omega)$ is given by

$$f_Z(\omega) = \frac{\sigma^2}{2\pi} \frac{\left|\theta\left(e^{-i\omega}\right)\right|^2}{\left|\phi\left(e^{-i\omega}\right)\right|^2}. \tag{7}$$

AR and MA spectra are special cases of this spectrum when $\theta(x) = 1$ and $\phi(x) = 1$, respectively. Therefore if the process {Z(t)} is an ARMA(p,q) process and is correctly fitted by the ARMA(p,q) model, then we have $S_{p,q}(\Theta) = \frac{\sigma^2}{2\pi}$, which is a spectral density of a white noise process.

Let {X(t)} be a weakly stationary Exponential process of order r, EX(r), and the spectral density $f_X(\omega\,;r)$ is given by

$$f_X(\omega\,;r) = \frac{1}{2\pi}\,\mathrm{Exp}\left[\alpha_0 + \sum_{k=1}^{r}\alpha_k \,\mathrm{Cos}\, k\omega\right]. \tag{8}$$

(See Nakatuka [6]). For example, when r = 2, its spectral density function has the following graphs when $\{\alpha_0, \alpha_1, \alpha_2\}=\{0, 0.5, -1.0\}$ and when $\{\alpha_0, \alpha_1, \alpha_2\}=\{0, -0.1, 0.7\}$ shown in Figure 2.

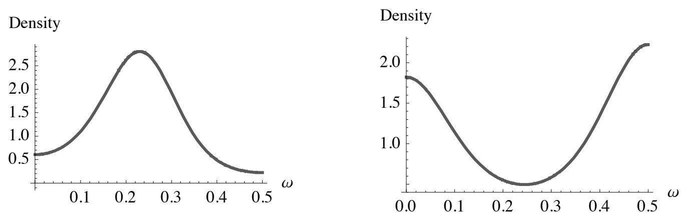

Figure. 2. Graphs of spectra of EX(2) processes

It is also seen that the auto-covariance function of the process EX(1) is $\mathrm{Cov}(X(t), X(t+h)) = I_h(a)$, the modified Bessel function of order h.

When we consider an ARMA(p,q) model fitting to this Ex(r) process {X(t)}, $S_{p,q,r}(\Theta)$ is expressed as

$$S_{p,q,r}(\Theta) = \int_{-1/2}^{1/2} \frac{\left|\prod_{k=1}^{p}[1-\phi_k \exp(-2\pi i\omega)]\right|^2}{\left|\prod_{j=1}^{q}[1-\theta_j \exp(-2\pi i\omega)]\right|^2} f_X(\omega\,;r)\, d\omega. \tag{9}$$

In this paper, we consider the case when an ARMA(1,1) model is fitted incorrectly to an EX(2) process {X(t)}; Here we set the ARMA(1,1) model parameters (x, y) in stead of (ϕ, θ). In this case, $S_{p,q,r}(\Theta)$ can be derived from (9), ignoring the constant term $\frac{\sigma^2}{2\pi}$, as the following expression.

Proposition 1

$$S_{1,1,2}(x, y) = S_{1,1,2}(x, y\,;\, a, b) = $$

$$= \frac{1}{\left(1-y^2\right)} \left(\left[2x E_1(a,b) + \left(1+x^2\right)E_0(a,b)\right] + 2\sum_{k=1}^{\infty} y^k \left\{x\,[E_{k-1}(a,b) + E_{k+1}(a,b)] + \left(1+x^2\right)E_k(a,b)\right\}\right), \tag{10}$$

where $E_k(a, b) = \sum_{j=-\infty}^{\infty} I_{2j+k}(a)\, I_{-j}(b)$, and $I_k(a)$ is the modified Bessel function of order k.

Remark: If r = 1 (or b = 0), then $E_k(a, b) = I_k(a)$ for all k, and thus we can derive that, for an ARMA(1,1) model fitting to EX(1) processes,

$$S_{1,1,1}(x, y) = S_{1,1,1}(x, y\,;\, a) =$$

$$\frac{\left[2\,x\; I_1(a) + \left(1 + x^2\right) I_0(a)\right] \;+\; 2\sum_{k=1}^{\infty} y^k \left\{x\left[\,I_{k-1}(a)\right) + \;I_{k+1}(a)\right] + \left(1 + x^2\right) I_k(a)\right\}}{\left(1 - y^2\right)}. \tag{11}$$

For the EX(2) process {X(t)}, we should note that Cov(X(t), X(t+h)) = $E_h(a, b)$.

The covariance function of the residuals {e(t)} of the ARMA(p,q) model fitting to the EX(r) process is defined by

$$R_{p,q,r}(h\,;\Theta) = E[e(t)\,e(t+h)]$$

$$= \int_{-1/2}^{1/2} \exp(-2\,\pi\mathrm{i}\;\mathrm{h}\omega)\;\frac{|\prod_{k=1}^{p}[1 - \phi_k \exp(-2\,\pi\mathrm{i}\omega)]\,|^2}{|\prod_{j=1}^{q}[1 - \theta_j \exp(-2\,\pi\mathrm{i}\omega)]\,|^2}\, f_Z(\omega)\, d\omega. \tag{12}$$

Remark. When h = 0 in (13), $R_{p,q,r}(h\,;\Theta) = S_{p,q,r}(\Theta)$.

In the case when an ARMA(1,1) model is fitted incorrectly to an EX(2) process {X(t)}, the covariance function of the residuals is evaluated by the following expression.

Proposition 2.

$$\begin{aligned} R_{1,1,2}(h\,;\, x, y) &= R_{1,1,2}(h\,;\, x, y\,;\, a, b) \\ &= \left(1 - y^2\right)^{-1}\Bigg(\Big[x\,(E_{h-1}(a, b) + E_{h+1}(a, b)) + \left(1 + x^2\right) E_h(a, b)\Big] + \\ &\sum_{k=1}^{\infty} y^k \Big\{x\,[E_{k+h-1}(a, b) + E_{k+h+1}(a, b) + E_{k-h-1}(a, b) + E_{k-h+1}(a, b)] + \\ &\left(1 + x^2\right)[E_{k-h}(a, b) + E_{k+h}(a, b)]\Big\}\Bigg), \end{aligned} \tag{13}$$

where $E_k(a, b) \;=\; \sum_{j=-\infty}^{\infty} I_{2\,j+k}(a)\, I_{-j}(b)$, and $I_k(a)$ is the modified Bessel function of order k.

It is seen from (13) that $R_{1,1,2}(0\,;\, x, y) \;=\; S_{1,1,2}(x, y)$.

3. Numerical results

If we fit the ARMA(1,1) model to a EX(2) process, the local minimum of the function $S_{1,1,2}(x, y)$ is not necessarily one. We show two examples having two locally minimal points of the function $S_{1,1,2}(x, y)$.

Example 1. We consider the MA(1) model fitting to an EX(2) process whose parameters are $a = 0$, $b = -1.5$. It is seen by Figure 3 that $S_{0,1,2}(0, y)$ has two locally minimal points at y=-0.55 and 0.55. Graphs of their spectral densities are also shown in Figure.3.

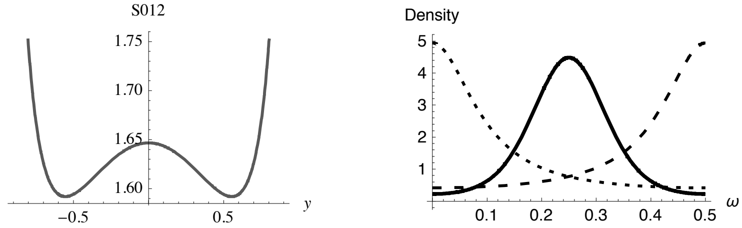

Figure 3. Graphs of $S_{0,1,2}(0, y)$ for an EX(2) process with a = 0 and b = -1.5 and spectra

Example 2. We consider the MA(1) model fitting to an EX(2) process whose parameters are a = -0.1, b = 0.7. The graph of the covariance function of the process is shown in Figure 4. It is seen that $S_{1,1,2}(x, y)$ has two locally minimal points, {0.7,-0.63} and {-0.81,0.74}. Their spectral densities are also shown in Figure.4.

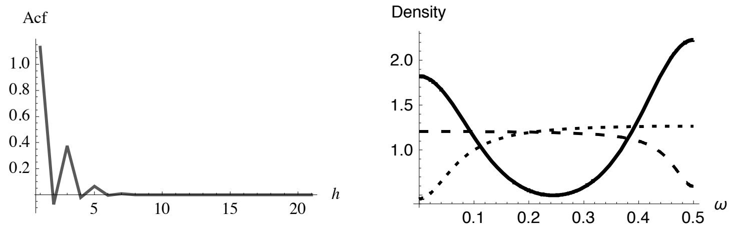

Figure 4. Graphs of covariance function of EX(2) process with a = -0.1 and b = 0.7 and spectra ,

4. Conclusion

In this paper, we have considered the misspecified ARMA(1,1) model fitting to EX(2) processes. We have evaluated the covariance function of the residuals {e(t)} of the ARMA(1,1) model fitting to the EX(2) process. Also, using these results, we have illustrated the examples that the local minimum of the function for the residual variance $S_{1,1,2}(x, y)$ is not necessarily one by the numerical analysis study. We know that it will be related to critical point theory and the behavior of degenerate critical points of the function of two variables in Catastrophe theory, considering the ARMA(1,1) quasi-likelihood function as a potential function with two external parameters a and b. On the misspecified MA(1) model fitting to AR(2) processes, it was already seen that the domain for AR(2) parameters on which the MA(1) quasi-likelihood function has more than one local maximum points is related to a cusp catastrophe (see [7]). Thus it will be a future work for us to investigate the conditions for EX(2) parameters on which ARMA(1,1) quasi-likelihood function has more than one local maximum points in the stationary and invertible parameter space.

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